Available online at www.jpsscientificpublications.com
Volume – 2; Issue - 5; Year – 2016; Page: 752 – 756
DOI: 10.21276.iajmr.2016.2.5.5

Indo – Asian Journal of Multidisciplinary Research (IAJMR)
ISSN: 2454-1370

ESTIMATION OF EXPECTED TIME TO SEROCONVERSION OF HIV INFECTED WHEN THE ANTIGENIC DIVERSITY THRESHOLD FOLLOWING THREE PARAMETER GENERALIZED EXPONENTIAL DISTRIBUTION

K. Siva*, C. Subramanian and R. Sathiyamoorthi,
Department of Statistics, Annamalai University, Annamalai Nagar - 608 002, Tamil Nadu, India.

Abstract

HIV infection and its progression to AIDS has become an important subject matter of study. The medical personnel spend much of the time and energy to find a cure from this infection. The mathematicians and statisticians pay their attention to develop stochastic models for the study of the spread of the infection and its progression. An interesting aspect of study is to develop models which could predict the time to serocoversion of HIV infected. In this paper a stochastic model is developed to find the expected time to seroconversion under the assumption that the antigenic diversity threshold is a random variable which follows Three Parameter Generalized Exponential Distribution. Basically the shock model and cumulative damage process is being used. Numerical illustration is also provided.

Key words: Antigenic Diversity, Threshold Seroconversion, Expected Time and Shock Model Approach.

1. Introduction

HIV infection and its progression to AIDS has become a matter of great concern to the people, government and medical experts. Since no medicine is made available for the eradication of the infection tile this date much of efforts are taken by the people as well as by the government to avoid the spread of this infection. One of the important aspects of the study and research is to formulate mathematical and stochastic models to portray the progression of this infection to the extent of seroconversion and then to the case of full blown AIDS. The antigenic diversity of the invading antigens and the accumulation of antigenic diversity due to successive contacts with the infected is an important aspect. Many research papers have been published by several authors in this connection. Several others have discussed stochastic models for estimation of the expected time to seroconversion Sathiyamoorthy and Kannan (1998) have discussed a stochastic model to estimate the expected time to seroconversion. Stilianakis *et al.* (1994) have discussed a model and the antigenic diversity threshold for AIDS. Tan (1991) has discussed on some stochastic models for HIV epidemic among homo sexual population. The concept of Genetralaized Exponential Distribution has been discussed by Gupta and Kundu (1999). Esary *et al.* (1973) have discussed the shock model and wear processes. Sathiyamoorthy (1980) has discussed a cumulative damage model with correlated inter arrival time of shocks. In this paper, a stochastic model to estimate the expected time to seroconversion due to successive contacts is derived, under the following assumptions.

Corresponding author*: **K. Siva
E. mail: statsivaphd@gmail.com
Received: 12.07.2016; *Revised*: 18.08.2016;
Accepted: 10.09.2016.

2. Assumptions

- A person is exposed to HIV infection at random time intervals and during every occasion of exposure there is a random amount of contribution to antigenic diversity of the in invading antigen.
- There is a particular random level of antigenic diversity beyond which the seroconversion of the infected takes place.
- The time intervals between successive contacts are of random character and hence it is a random variable.

3. Notations

X_i a random variable which denotes the random amount of contribution antigen diversity, i= 1, 2, 3 … n. X_i has pdf g (.).

Y is the threshold level of antigenic diversity with pdf h (.) and cdf H (.).

U the time interval between successive contacts denoted as a random variable which has pdf f (.). The time interval and between successive contact are identically independently distributed. The c.d.f is F (.).

S (t) the survival function which gives the probability that the total contribution to antigenic diversity does not cross the threshold level.

L (t) = 1 – S (t).

$L^*(s)$ = Laplace transform of L (t).

4. Model Description

It is assumed that the threshold level was distributed as a three parameter generalized exponential distribution. The cumulative distribution function is

$$H\ (\ y: \alpha\,, \lambda\,, \theta) = (1 - e^{-\lambda(x-\theta)})^{\alpha\,;} \quad \boldsymbol{x} > \theta\,, \quad \alpha\,, \lambda > 0\,.$$

Probability density function is

$$h\ (y: \alpha\,, \lambda\,, \theta) = \alpha\,\lambda\,(1 - e^{-\lambda(x-\theta)})^{\alpha-1}\, e^{-\lambda(x-\theta)\,;} \quad \boldsymbol{x} > \theta\,, \quad \alpha\,, \lambda > 0\,.$$

$$\overline{H}\ (y) = 1 - H\ (y)$$

$$= 1 - (1 - e^{-\lambda(x-\theta)})^{\alpha}$$

Taking the shape parameter as $\alpha = 1$

Put $\alpha = 1$

$$= 1 - (1 - e^{-\lambda(x-\theta)})$$

$$= 1 - 1 + e^{-\lambda(x-\theta)}$$

$$\overline{H}\ (y) = e^{-\lambda(x-\theta)}$$

$$P(\textstyle\sum_{n=k}^{n} x_i < y) = \int_0^{\infty} g_k(x)\overline{H}(y)dx$$

$$= \int_0^{\infty} g_k(x)\ e^{-\lambda(x-\theta)}\,dx$$

$$= \int_0^{\infty} g_k(x)\ e^{-\lambda x+\lambda\theta}\ dx$$

$$= e^{\lambda\theta} \int_0^{\infty} g_k(x)\ e^{-\lambda x}\ dx$$

$$P(\textstyle\sum_{n=k}^{n} x_i < y) = e^{\lambda\theta}\ [\,g^*(\lambda)\,]^k$$

The survival function gives the probability that the cumulative antigenic diversity does not cross the threshold level before time ‘t’.

Hence, $S(t) = P(T > t)$

$$= \sum_{k=0}^{\infty} P\{\,there\ are\ exactly\ k\ contacts\ (0,t]\}$$
$$* P\{\,the\ cumulative\ antigenic\ diversity\ does\ not\ cro$$

It is also known from renewal theory that

V (t) = P (exactly k contacts or exposures in (0, t))
$=F_k(t) - F_{k+1}(t)$ with $F_0(t) = 1$

$$P(T > t) = \textstyle\sum_{n=k}^{\infty} V_k(t) P(\sum_{i=1}^{n} x_i < y)$$

$$= \textstyle\sum_{n=k}^{\infty} [F_k(t) - F_{k+1}(t)]\ e^{\lambda\theta}\,[\,g * (\,\lambda)\,]^k$$

Now, $L(t) = 1 - S(t)$

$$L(t) = 1 - \sum_{k=0}^{\infty}[F_k(t) -$$
Fk+1t eλθ g∗ λ k

$$= \left(1 - e^{\lambda\theta}\right)\left[\sum_{k=0}^{\infty} F_k(t) - F_{k+1}(t)\right] [\, g * (\,\lambda)\,]^k$$

On simplification we get

$$L(t) = \left(1 - e^{\lambda\theta}\right) \{\textstyle\sum_{k=0}^{\infty}[F_k(t) - F_{k+1}(t)]\} \; [\, g * (\,\lambda)\,]^k$$

$$= \left(1 - e^{\lambda\theta}\right)[F_0(t) - F_1(t)] \;\; g^*(\lambda)^0 + [F_1(t) - F_2(t)] \;\; g^*(\lambda)^1 + [F_2(t) - F_3(t)]\, g^*(\lambda)^2 + \;\ldots$$

$$= \left(1 - e^{\lambda\theta}\right)\left[1 + \left(1 - g^*(\lambda)\right)F_1(t)\right] - \left[\left(1 - g^*(\lambda)\right)F_2(t)\right]$$

$$L(t) = \left(1 - e^{\lambda\theta}\right) \{\textstyle\sum_{k=0}^{\infty}[F_k(t) - F_{k+1}(t)]\} \, [g^*(\lambda)]^k$$

.

$$= \left(1 - e^{\lambda\theta}\right)\{[F_0(t) - F_1(t)]\}\, g^*(\lambda)^0 + [F_1(t) - F_2(t)]g^*(\lambda)^1 + [F_2(t) - F_3(t)]g^*(\lambda)^2 + \;\ldots$$

$$= \left(1 - e^{\lambda\theta}\right)\left[1 + \left(1 - g^*(\lambda)\right)F_1(t)\right] - \left[\left(1 - g^*(\lambda)\right)F_2(t)\right]$$

$$= \left(1 - e^{\lambda\theta}\right) + e^{\lambda\theta}\left(1 - g^*(\lambda)\right)F_1(t)\sum_{K=1}^{\infty} F_k(t)\;\; g^*(\lambda)^{k-1}$$

$$= \left(1 - e^{\lambda\theta}\right) + e^{\lambda\theta}\,[1 -$$
g∗λ K=1∞Fk(t) g∗λk−1

Taking Laplace transformation of L (t) we get

$$l^*(s) = \frac{1}{s}\left(1 - e^{\lambda\theta}\right) + e^{\lambda\theta}\,[1 - g^*(\lambda)] \sum_{K=1}^{\infty} F_k^*(s)\;\; g^*(\lambda)^{k-1}$$

$$l^*(s) = \frac{s}{s}\left(1 - e^{\lambda\theta}\right) + e^{\lambda\theta}\,[1 - g^*(\lambda)]\, f^*(s) \sum_{K=1}^{\infty}\left(f^*(s)\right)\; g^*(\lambda)^{k-1}$$

$$l^*(s) = \left(1 - e^{\lambda\theta}\right) + e^{\lambda\theta}\,[1 - g^*(\lambda)]\, f^*(s)\, \left[1 - \left(f^*(s)\right) g^*(\lambda)\right]^{-1}$$

$$l^*(s) = \left(1 - e^{\lambda\theta}\right) + \frac{e^{\lambda\theta}\,[1-g^*(\lambda)]\, f^*(s)}{\left[1-\left(f^*(s)\right) g^*(\lambda)\right]}$$

Let the random variable U denoting inter arrival times between successive exposures follow exponential distribution with parameter ‘c’. Now $f^*(s) = \left(\frac{c}{c+s}\right)$. Substituting $f^*(s)$ in the above equation we get

$$l^*(s) = \left(1 - e^{\lambda\theta}\right) + \frac{e^{\lambda\theta}\,[1-g^*(\lambda)]\left[\frac{c}{c+s}\right]}{\left[1-\left(\frac{c}{c+s}\right) g^*(\lambda)\right]}$$

$$= \left(1 - e^{\lambda\theta}\right) + \frac{e^{\lambda\theta}\,[1-g^*(\lambda)]\left[\frac{c}{c+s}\right]}{\left[\frac{c}{c+s}\right]\left[\left(\frac{c+s}{c}\right) - g^*(\lambda)\right]}$$

$$l^*(s) = \left(1 - e^{\lambda\theta}\right) + e^{\lambda\theta}\,[1 - g^*(\lambda)]\left[\left(\frac{c+s}{c}\right) - g^*(\lambda)\right]^{-1}$$

$$\frac{d}{ds}l^*(s) = \frac{d}{ds}\left(1 - e^{\lambda\theta}\right) + e^{\lambda\theta}\,[1 - g^*(\lambda)]\left[\left(\frac{c+s}{c}\right) - g^*(\lambda)\right]^{-1}$$

$$= 0 + e^{\lambda\theta}\,[1 - g^*(\lambda)]\frac{d}{ds}\left[\left(\frac{c+s}{c}\right) - g^*(\lambda)\right]^{-1}$$

$$= e^{\lambda\theta}\,[1 - g^*(\lambda)](-1)\left[\left(\frac{c+s}{c}\right) - g^*(\lambda)\right]^{-2} .1$$

$$= - e^{\lambda\theta}[1 - g^*(\lambda)]\left[\left(\frac{c+s}{c}\right) - \right.$$

*g*λ−2*

$$E(T) = \frac{-d}{ds} l^*(s) \ |_{s=0}$$

$$= - e^{\lambda\theta}[1 -$$

*g*λ1− g*λ−2× 1c*

$$= \frac{e^{\lambda\theta}}{c}[1 - g^*(\lambda)]^{-1}$$

$$E(T) = \frac{e^{\lambda\theta}}{c[1 - g^*(\lambda)]}$$

$$g(.) \sim \exp(\mu), g^*(\lambda) = \frac{\mu}{\mu+\lambda}$$

On simplification we get

$$E(T) = \frac{e^{\lambda\theta}}{c}\left[1 - \frac{\mu}{\mu+\lambda}\right]^{-1}$$

$$= \frac{e^{\lambda\theta}}{c}\left[\frac{\mu+\lambda-\mu}{\mu+\lambda}\right]^{-1} = \frac{e^{\lambda\theta}}{c}\left[\frac{\lambda}{\mu+\lambda}\right]^{-1}$$

$$E(T) = \frac{e^{\lambda\theta}}{c}\left[\frac{\mu+\lambda}{\lambda}\right]$$

$$E(T^2) = \frac{-d^2}{ds^2} l^*(s) =$$

$$\frac{d}{ds}\left[- e^{\lambda\theta}\left(1 - g^*(\lambda)\right)\left(\left(\frac{c+s}{c}\right) - g^*(\lambda)\right)^{-2}\right]$$

$$= - e^{\lambda\theta}\left(1 - g^*(\lambda)\right)\frac{d}{ds}\left[\left(\frac{c+s}{c}\right) - g^*(\lambda)\right]^{-2}$$

$$= \frac{-e^{\lambda\theta}}{c}\left(1 - g^*(\lambda)\right)(-2)\left[\left(\frac{c+s}{c}\right) - g^*(\lambda)\right]^{-3}.1$$

$$\frac{-d^2}{ds^2} l^*(s) \ |_{s=0} = 2\ e^{\lambda\theta}\left(1 - g^*(\lambda)\right)\left(1 - \right.$$

*g*λ−3× 1c*

$$E(T^2) = \frac{2\, e^{\lambda\theta}\left(1 - g^*(\lambda)\right)^{-2}}{c^2}$$

$$E(T^2) = \frac{2\, e^{\lambda\theta}}{c^2}\left[\frac{1}{\left(1 - g^*(\lambda)\right)^2}\right]$$

$$= \frac{2\, e^{\lambda\theta}}{c^2}\left[\frac{1}{\left(1 - \frac{?}{? + \lambda}\right)^2}\right]$$

$$= \frac{2\, e^{\lambda\theta}}{c^2}\left[\frac{1}{\left(\frac{? + \lambda - ?}{? + \lambda}\right)^2}\right]$$

$$E(T^2) = \frac{2\, e^{\lambda\theta}}{c^2}\left[\frac{1}{\left(\frac{\lambda}{? + \lambda}\right)^2}\right]$$

$$= \frac{2\, e^{\lambda\theta}}{c^2}\left[\frac{? + \lambda}{\lambda}\right]^2$$

$$V(T) = E(T^2) - [E(T)]^2$$

$$= \frac{2\, e^{\lambda\theta}}{c^2}\left[\frac{?+\lambda}{\lambda}\right]^2 - \left[\frac{e^{\lambda\theta}}{c}\left[\frac{?+\lambda}{\lambda}\right]\right]^2$$

$$V(T) = \frac{e^{\lambda\theta}\,[?+\lambda]^2}{c^2\lambda^2}\left(2 - e^{\lambda\theta}\right)$$ and $$E(T) = \frac{e^{\lambda\theta}}{c}\left[\frac{?+\lambda}{\lambda}\right]$$

5. Numerical illustration and Conclusions

Table 1: λ = 0.2 μ = 0.4 c = 0.6

θ	E(T)	V(T)
0.2	5.20	24.96
0.4	5.42	24.84
0.6	5.64	24.57
0.8	5.87	24.28

a) If the value of ‘θ' which is the parameter of the indicating threshold distribution increases then there is an increase value of E (T) and also V(T) increases to a very small extant.

b) If the values of the parameter ‘λ’ increases E (T) decreases and V (T) also shows a decrease.

c) If the parameter ‘c’ of inter arrival time between successive contacts shows an increase then E (U) $= \frac{1}{c}$ decreases. Therefore the inter arrival time between successive contacts is smaller and hence the value of E (T) and V (T) decrease.

d) If the value of parameter $'\mu'$ increases then E (T) increases. This is due to fact that the random variable X_i which denotes the contribution antigenic diversity follows exponential distribution and so E(X) = $\frac{1}{\mu}$ which decreases. Therefore there is less of contribution of to antigenic diversity and so E (T) increases000.

6. Reference

1) Easary, J. D., A. W. Marshall and F. Proschan. 1973. Shock models and wear Processes. *Annals of Probability*, 1(4): 627 - 649.
2) Gupta, R. D and Kundu, D. 1999. Generalized Exponential Distribution. *Austral Statistics*, 41 (2): 173 - 188.
3) Sathiyamoorthi, R. 1980. Cumulative damage model with correlated inter arrival time of shocks. *IEEE Transactions on Reliability*, R-29, No. 3.
4) Sathiyamoorthi, R and R. Kannan. 1998. On the time to seroconversion of HIV Patients under correlated Intercontact times. *Pure and applied Mathematika Sciences*, 1 (2): 75 - 87.
5) Stilianakis, N. I., D. Schenzle and K. Dietz. 1994. On the Antigenic Diversity Threshold Model for AIDS. *Mathematical Biosciences*, 121: 235 - 247.
6) Tan, W. Y. 1991. Stochastic Models of HIV Epidemic in Homosexual Populations – the effects of mixing patterns. *Mathematical and Computer Modeling*, 15 (12): 37 - 65.